13. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider the Landau-Ginzburg free energy functional for a magnet with magnetization M:

$$F(M) = \frac{1}{2}rM^2 + uM^4 - hM$$

M takes values $M \in [-\infty, \infty]$. (The rotational symmetry of the magnet is broken by the crystal so that M is a scalar, not a vector.) $r = a(T - T_c)$, u is only weakly dependent on T, and h is the magnetic field. We will make the mean-field approximation that M is equal to the value which minimizes $F(M)$, and $F(M)$ is given by its minimum value.

(a) For $T > T_c$ and $h = 0$, what value of M minimizes F? For $T < T_c$ and $h = 0$, what value of M minimizes F?

(b) For $h = 0$, the specific heat takes the asymptotic form $C \sim |T - T_c|^{-\alpha}$ as $T \to T_c$. What is α?

(c) At $T = T_c$, $M \sim h^{\delta}$. What is δ?

(units $\Rightarrow$ specific free energy) $f(T) = F(T, M_{min})$

(h = ext. B-field, not H-field)

a) $h = 0 \qquad F(T,M) = \frac{1}{2} r M^2 + u M^4 \qquad r = a(T - T_c)$

$F' = rM + 4uM^3 = M(r + 4uM^2) = 0 \quad \Rightarrow \quad M = 0,\ \pm\sqrt{-\frac{r}{4u}} \equiv \pm A$

$F'' = r + 12uM^2 \qquad F''(0) = r \qquad F''(\pm A) = r + 12u\left(-\frac{r}{4u}\right) = -2r$

minimize $F \Rightarrow F' = 0,\ F'' > 0$

T	a	u	r	$M=0$	$M=\pm A$	F''	M_{min}	Conclusion
$T > T_c$	$\boxed{a > 0}$	$\boxed{u > 0}$	+	min	complex	+	0	correct *
		$u < 0$	+	min	max.s	±	$0, \pm\infty$	(probably not)
	$a < 0$	$u > 0$	−	max	min.s	±	$\pm A$	
		$u < 0$	−	max	complex	−	$\pm\infty$	(not physical)
$T < T_c$	$\boxed{a > 0}$	$\boxed{u > 0}$	−	max	min.s	±	$\pm A$	
		$u < 0$	−	max	complex	−	$\pm\infty$	×
	$a < 0$	$u > 0$	+	min	complex	+	0	
		$u < 0$	+	min	max.s	±	$0, \pm\infty$	×

* ($T > T_c \Rightarrow M = 0$)

Assuming $a > 0$ and $u > 0$ (which makes the most sense above)

$M = 0$ minimizes $F(T,M)$ for $T > T_c$ and

$M = \pm A = \pm\sqrt{-\frac{r}{4u}} = \pm\frac{1}{2}\sqrt{\frac{a}{u}|T - T_c|}$ minimizes F for $T < T_c$.

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b) $h = 0$

$$C = \frac{dQ}{dT} = T\frac{ds}{dT}$$

$h=0 \Rightarrow$ $du = T\,ds$, $u = u(s)$ $\quad f = u - Ts \quad$ $df = -s\,dT \Rightarrow -s = \frac{df}{dT}$, $f = f(T)$

$$c = T\frac{ds}{dT} = T\frac{d}{dT}\left(-\frac{df}{dT}\right) = -T\frac{d^2f}{dT^2}$$

$$f(T) = F(T, M_{min}) = \begin{cases} 0 & \text{if } T > T_c \\ \frac{1}{2}a(T-T_c)\frac{1}{4}\frac{a}{u}|T-T_c| + u\frac{1}{16}\frac{a^2}{u^2}|T-T_c|^2 & \end{cases}$$

$$= -\frac{1}{8}\frac{a^2}{u}|T-T_c|^2 + \frac{1}{16}\frac{a^2}{u}|T-T_c|^2$$

$$= -\frac{1}{16}\frac{a^2}{u}|T-T_c|^2 \quad \text{if } T < T_c$$

$$\Rightarrow c = -T\frac{d^2f}{dT^2} = -T\frac{d}{dT}\left[-\frac{1}{8}\frac{a^2}{u}|T-T_c|\right] = -T\left[-\frac{1}{8}\frac{a^2}{u}\right] = \frac{1}{8}\frac{a^2}{u}T$$

$$= \frac{1}{8}\frac{a^2}{u}(T-T_c) + \frac{1}{8}\frac{a^2}{u}T_c = -\frac{1}{8}\frac{a^2}{u}|T-T_c| + \frac{1}{8}\frac{a^2}{u}T_c$$

for $T < T_c$ $\quad$ ($c = 0$ for $T \to T_c$ from above)

As $T \to T_c$ from below $|T-T_c| \ll 1$, so the term $\frac{1}{8}\frac{a^2}{u}T_c$ dominates.

$$\Rightarrow \quad c \sim |T-T_c|^0 \quad \Rightarrow \quad \alpha = 0$$

c) $T = T_c \Rightarrow r = 0$ $\quad$ $F(T,M) = uM^4 - hM$

$$F' = 4uM^3 - h = 0 \quad \Rightarrow \quad M = \left(\frac{h}{4u}\right)^{1/3}$$

$$F'' = 12uM^2 > 0$$

$\therefore \quad M \sim h^{1/3}$

$$\delta = \frac{1}{3}$$