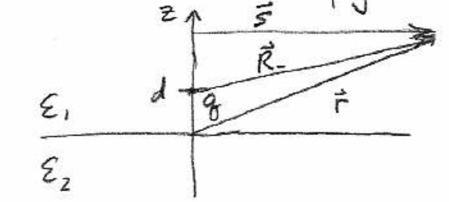
10. *Electricity and Magnetism* (Fall 2004)

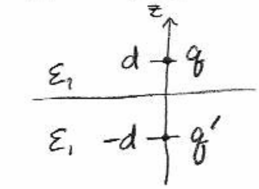
The upper half-space is filled with a material of permittivity ϵ_1, while the lower half space is filled with a different material with permittivity ϵ_2. Their interface is located at the $z = 0$ plane. A point charge q is located at $\mathbf{r}_q = d\hat{\mathbf{z}}$ on the z-axis in medium 1. Find the electrostatic potential everywhere.

cf. Jackson pg 254 (I cannot yet explain why this works...)

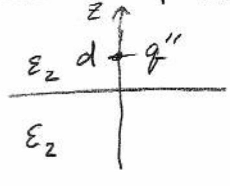


We use a method analogous to the method of images, using two different physical setups when solving for the potential in the...

... Upper space



... Lower space



$$\Phi(\vec{r}) = \Phi(s,z) = \begin{cases} \Phi_1(s,z) = \frac{1}{4\pi\varepsilon_1}\left[\frac{q}{R_-} + \frac{q'}{R_+}\right] \\ \Phi_2(s,z) = \frac{1}{4\pi\varepsilon_2}\frac{q''}{R_-} \end{cases}$$

where $R_\pm = \sqrt{s^2 + (z \pm d)^2}$

B.C.s: $(\Delta D)^\perp = \Sigma^f = 0 \qquad (\Delta \vec{E})^\parallel = \vec{0}$

$$\Rightarrow \quad \varepsilon_1 \vec{\nabla}\Phi_1 \cdot \hat{z}\Big|_{z=0} = \varepsilon_2 \vec{\nabla}\Phi_2 \cdot \hat{z}\Big|_{z=0} \quad \Rightarrow \quad \varepsilon_1 \partial_z \Phi_1\Big|_{z=0} = \varepsilon_2 \partial_z \Phi\Big|_{z=0}$$

$$\Rightarrow \varepsilon_1 \frac{\cancel{1}}{\cancel{4\pi}\varepsilon_1}\left[q\,\partial_z\left(\frac{1}{R_-}\right) + q'\,\partial_z\left(\frac{1}{R_+}\right)\right]\Big|_{z=0} = \varepsilon_2 \frac{\cancel{1}}{\cancel{4\pi}\varepsilon_2} q''\,\partial_z\left(\frac{1}{R_-}\right)\Big|_{z=0}$$

and $\partial_z\left(\frac{1}{R_\pm}\right)\Big|_{z=0} = \frac{1}{2}\left[s^2 + (z\pm d)^2\right]^{-1/2} 2(z \pm d)\Big|_{z=0} = \pm d\left[s^2 + d^2\right]^{-1/2}$

so $\partial_z\left(\frac{1}{R_+}\right)\Big|_{z=0} = -\partial_z\left(\frac{1}{R_-}\right)\Big|_{z=0}$

$$\Rightarrow \quad q - q' = q'' \quad ①$$

and $(\vec{\nabla}\Phi_1)^{s\phi}\Big|_{z=0} = (\vec{\nabla}\Phi_2)^{s\phi}\Big|_{z=0} \Rightarrow$ (no ϕ-dependence) $\partial_s \Phi_1\Big|_{z=0} = \partial_s \Phi_2\Big|_{z=0}$

$$\Rightarrow \frac{\cancel{1}}{4\pi\varepsilon_1}\left[q\,\partial_s\left(\frac{1}{R_-}\right) + q'\,\partial_s\left(\frac{1}{R_+}\right)\right]\Big|_{z=0} = \frac{\cancel{1}}{4\pi\varepsilon_2} q''\,\partial_s\left(\frac{1}{R_-}\right)\Big|_{z=0}$$

and $\partial_s\left(\frac{1}{R_\pm}\right)\Big|_{z=0} = \frac{1}{2}\left[s^2 + (z\pm d)^2\right]^{-1/2} 2s$ so $\partial_s\left(\frac{1}{R_+}\right)\Big|_{z=0} = \partial_s\left(\frac{1}{R_-}\right)\Big|_{z=0}$

$$\Rightarrow \quad q + q' = \frac{\varepsilon_1}{\varepsilon_2} q'' \quad ②$$

Thus ①+② $\Rightarrow 2q = (1 + \varepsilon_1/\varepsilon_2) q'' \Rightarrow q'' = \dfrac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} q \qquad \bar{\varepsilon} \equiv \dfrac{\varepsilon_1 + \varepsilon_2}{2}$

② $- \frac{\varepsilon_1}{\varepsilon_2}$① $\Rightarrow \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right) q + \left(1 + \frac{\varepsilon_1}{\varepsilon_2}\right) q' = 0 \qquad \Rightarrow q' = \dfrac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} q$

$$\Rightarrow \Phi_1(s,z) = \frac{1}{4\pi\varepsilon_1}\frac{q}{\sqrt{s^2 + (z-d)^2}} + \frac{1}{4\pi\varepsilon_1}\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)\frac{q}{\sqrt{s^2 + (z+d)^2}} \qquad \Phi_2(s,z) = \frac{1}{4\pi\bar{\varepsilon}}\frac{q}{\sqrt{s^2 + (z-d)^2}}$$