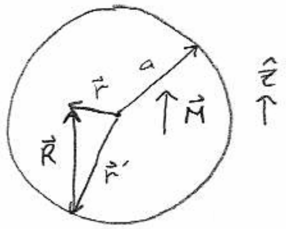
8. *Electricity and Magnetism* (Fall 2004)

Consider a sphere of radius a with uniform magnetization **M**, pointing in the z-direction. What are the magnetic induction **B** and magnetic field **H** inside the sphere?



$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

$$\vec{\nabla}\times\vec{H} - \cancel{\partial_t \vec{D}}^{\,\vec{0}} = \vec{J}^f = \vec{0} \quad \Rightarrow \quad \vec{H} = -\vec{\nabla}\Phi_m$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} = \rho_m$$

$$\Rightarrow \quad \nabla^2 \Phi_m = -\rho_m = \vec{\nabla}\cdot\vec{M} \qquad \text{and} \qquad \Sigma_m = -\Delta M\cdot\hat{r} = M\cos\theta$$

"surface magnetic charge"

$$\Rightarrow \quad \Phi_m = \frac{1}{4\pi}\int_{R^3} \frac{\rho_m\, dv'}{R} \qquad \text{where } \rho_m = \rho_{m,\text{interior}} + \Sigma_m\,\delta(r-a)$$

and $\rho_{m,\text{interior}} = -\vec{\nabla}\cdot\vec{M}_{\text{interior}} = 0$

since $\vec{M}$ is constant in the interior

$$= \frac{1}{4\pi}\oint_S \frac{\Sigma_m\, a^2\, d\Omega'}{R} = \frac{Ma^2}{4\pi}\oint_S \frac{\cos\theta'\, d\Omega'}{R} \qquad \text{where } Y_{10}(\theta',\phi') = C\cos\theta'$$

$$= \frac{Ma^2}{4\pi}\oint_S d\Omega'\left[\frac{1}{C}Y_{10}(\theta',\phi')\right]\sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y_{\ell m}(\theta,\phi)\,Y^*_{\ell m}(\theta',\phi')$$

$$= \frac{Ma^2}{4\pi}\frac{1}{C}\sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{r^{\ell}}{a^{\ell+1}}Y_{\ell m}(\theta,\phi)\,\delta_{1\ell}\,\delta_{0m}$$

$$= \frac{M\cancel{a^2}}{\cancel{4\pi}}\frac{\cancel{4\pi}}{2+1}\frac{r^1}{\cancel{a^{1+1}}}\left(\frac{1}{C}Y_{10}(\theta,\phi)\right) = \frac{1}{3}Mr\cos\theta = \frac{1}{3}Mz$$

$$\vec{H} = -\vec{\nabla}\Phi_m = -\partial_z\Phi_m(z)\,\hat{z} = -\frac{1}{3}M\hat{z} = -\frac{1}{3}\vec{M}$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0\left(-\frac{1}{3}+1\right)\vec{M} = \frac{2}{3}\mu_0\vec{M}$$