3. *Quantum Mechanics* (Fall 2004)

A positron has the same mass m as the electron, but the opposite charge. Consider a set of states containing one electron and one positron. A complete set of these states can be labeled $|\mathbf{r}_+, \mathbf{r}_-\rangle$, where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the positions of the positron and electron, respectively. Normalize these states so that

$$\langle \mathbf{r}_+, \mathbf{r}_- | \mathbf{r}'_+, \mathbf{r}'_-\rangle = \delta_3(\mathbf{r}'_+ - \mathbf{r}_+)\, \delta_3(\mathbf{r}'_- - \mathbf{r}_-)$$

Then if the system is in any state $|\psi\rangle$, the wave function is

$$\psi(\mathbf{r}_+, \mathbf{r}_-) = \langle \mathbf{r}_+, \mathbf{r}_- | \psi\rangle$$

In this problem ignore spin.

(a) In terms of $\psi(\mathbf{r}_+, \mathbf{r}_-)$, what is the probability that at least one of the two particles is farther than a distance b from the origin?

(b) Write down the Hamiltonian for this electron-positron system, including the electrostatic (Coulomb) interactions between the two particles.

(c) Let $\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-$ and $\mathbf{R} = \frac{1}{2}(\mathbf{r}_+ + \mathbf{r}_-)$. Write the Hamiltonian in terms of the new coordinates and their canonically conjugate momenta $\mathbf{p}$ and $\mathbf{P}$.

(d) The bound electron-positron system is called *positronium*. For states with zero total momentum, write a formula for the possible negative values of the energy[1]. What is the approximate numerical value, in electron volts, of the ground state energy?

(e) Define the *charge conjugation* operator C on this system by

$$C\,|\mathbf{r}_+, \mathbf{r}_-\rangle = |\mathbf{r}_-, \mathbf{r}_+\rangle$$

Show that C commutes with the Hamiltonian. What is the eigenvalue of C on the state of lowest energy?

a) At least one particle farther than $r=b$

$\Rightarrow$ Not (both particles closer than $r=b$) $V_b \equiv$ sphere, radius b, at origin

$$P = 1 - \int_{V_b}\int_{V_b} |\Psi(\vec r_+, \vec r_-)|^2\, dv_1\, dv_2 = 1 - \int\int_{\text{all angles}} \int_0^b\int_0^b |\psi(\vec r_+, \vec r_-)|^2 r_+^2 dr_+\, r_-^2 dr_-\, d\Omega_+ d\Omega_-$$

b) $$H = \frac{\vec p_+^{\,2}}{2m} + \frac{\vec p_-^{\,2}}{2m} - K\frac{e^2}{|\vec r_+ - \vec r_-|}$$

c) $$H = \frac{\vec P^2}{2M} + \frac{\vec p^{\,2}}{2\mu} - K\frac{e^2}{r} \qquad M = 2m,\ \mu = \tfrac{1}{2}m$$

d) $$E_n = -\frac{\mu K^2 e^4}{2\hbar^2 n^2} = -\frac{m K^2 e^4}{4\hbar^2}\frac{1}{n^2} \approx -\frac{1}{2}(13.6\text{ eV})\frac{1}{n^2} \qquad E_1 \approx (6.8\text{ eV})$$

e) Since $\hat C^2 = \hat 1$, $\hat C \hat H \hat C = \hat H \Leftrightarrow [\hat C, \hat H] = \hat\emptyset$

$$\hat C \hat{\vec r}\, \hat C |\vec r_+, \vec r_-\rangle = \hat C \hat{\vec r}\, |\vec r_-, \vec r_+\rangle = \hat C(\hat{\vec r}_+ - \hat{\vec r}_-)|\vec r_-, \vec r_+\rangle = \hat C(\vec r_- - \vec r_+)|\vec r_-, \vec r_+\rangle$$

$$= (\vec r_- - \vec r_+)\hat C|\vec r_-, \vec r_+\rangle = -(\vec r_+ - \vec r_-)|\vec r_+, \vec r_-\rangle = -\vec r\,|\vec r_+, \vec r_-\rangle = -\hat{\vec r}\,|\vec r_+, \vec r_-\rangle$$

$\hat C \hat{\vec r} \hat C = -\hat{\vec r}$ for a complete basis $\Rightarrow$ $\hat C \hat{\vec r} \hat C = -\hat{\vec r}$ (for all state kets)

[1] Write your answer in terms of m, e^2 or α, $\hbar$, c, the Bohr radius, etc. You may use units in which $\hbar = c = 1$.

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e) (continued)

$$\hat{C}\hat{\vec{p}}\hat{C}\,|\vec{r}_+,\vec{r}_-\rangle = \hat{C}\,(\hat{\vec{p}}_+ - \hat{\vec{p}}_-)|\vec{r}_-,\vec{r}_+\rangle = \hat{C}\,(-i\hbar)(\vec{\nabla}_- - \vec{\nabla}_+)|\vec{r}_-,\vec{r}_+\rangle$$

$$= (-i\hbar)(\vec{\nabla}_- - \vec{\nabla}_+)\hat{C}\,|\vec{r}_-,\vec{r}_+\rangle = -(-i\hbar)(\vec{\nabla}_+ - \vec{\nabla}_-)|\vec{r}_+,\vec{r}_-\rangle$$

$$= -(\hat{\vec{p}}_+ - \hat{\vec{p}}_-)|\vec{r}_+ - \vec{r}_-\rangle = -\hat{\vec{p}}\,|\vec{r}_+,\vec{r}_-\rangle \qquad \Rightarrow \quad \hat{C}\hat{\vec{p}}\hat{C} = -\hat{\vec{p}}$$

$$\hat{C}\hat{\vec{P}}\hat{C} = \hat{C}\,\tfrac{1}{2}(\hat{\vec{p}}_+ + \hat{\vec{p}}_-)\hat{C} = \tfrac{1}{2}(\hat{\vec{p}}_- + \hat{\vec{p}}_+) = \hat{\vec{P}} \qquad \Rightarrow \quad \hat{C}\hat{\vec{P}}\hat{C} = \hat{\vec{P}}$$

$$\Rightarrow \quad \hat{C}\hat{H}\hat{C} = \hat{C}\left[\frac{\hat{\vec{P}}^2}{2M} + \frac{\hat{\vec{p}}^2}{2\mu} - \frac{Ke^2}{|\vec{r}|}\right]\hat{C} = \left[\frac{(\hat{\vec{P}})^2}{2M} + \frac{(-\hat{\vec{p}})^2}{2\mu} - \frac{Ke^2}{|-\hat{\vec{r}}|}\right]$$

$$= \left[\frac{\hat{\vec{P}}^2}{2M} + \frac{\hat{\vec{p}}^2}{2\mu} - \frac{Ke^2}{|\hat{\vec{r}}|}\right] = \hat{H}$$

$$\Rightarrow \quad [\hat{C},\hat{H}] = \hat{\emptyset} \qquad \checkmark$$

Since the lowest energy eigenstate is spherically symmetric with respect to $\vec{r}$ and $\hat{C}$ acts as the spatial parity operator with respect to $\vec{r}$, the lowest energy eigenstate is even in $\vec{r}$ and is an eigenstate of $\hat{C}$ with eigenvalue $+1$.