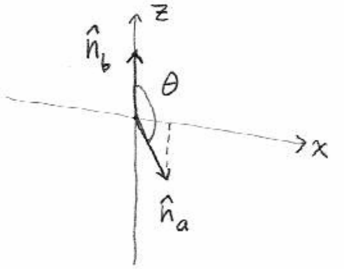
1. *Quantum Mechanics* (Fall 2004)

Two spin-half particles are in a state with total spin zero. Let $\hat{\mathbf{n}}_a$ and $\hat{\mathbf{n}}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{\mathbf{n}}_a$ and the spin of the second along $\hat{\mathbf{n}}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\mathbf{s}_a\cdot\hat{\mathbf{n}}_a\,\mathbf{s}_b\cdot\hat{\mathbf{n}}_b|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

Pick axes:



Note:
$$\sigma_z|\pm\rangle = \pm|\pm\rangle$$
$$\sigma_x|\pm\rangle = |\mp\rangle$$
$$\sigma_y|\pm\rangle = \pm i|\mp\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$\langle\psi|\vec{s}_a\cdot\hat{n}_a\,\vec{s}_b\cdot\hat{n}_b|\psi\rangle = \frac{\hbar^2}{4}\langle\psi|\vec{\sigma}_a\cdot\hat{n}_a\,\vec{\sigma}_b\cdot\hat{n}_b|\psi\rangle$$

$$= \frac{\hbar^2}{4}\langle\psi|(\cos\theta\,\sigma_{az} + \sin\theta\,\sigma_{ax})\sigma_{bz}\frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

$$= \frac{\hbar^2}{4}\frac{1}{2}\left[\langle+-| - \langle-+|\right]\Big[(+\cos\theta|+-\rangle + \sin\theta\cancel{|--\rangle})(-1)$$
$$-(-\cos\theta|-+\rangle + \sin\theta\cancel{|++\rangle})(+1)\Big]$$

$$= \frac{\hbar^2}{4}\frac{1}{2}\left[-\cos\theta\langle+-|+-\rangle - \cos\theta\langle-+|-+\rangle\right]$$

$$= -\frac{\hbar^2}{4}\cos\theta$$